Math 245C Lecture 3 Notes

Daniel Raban

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1 Cutoff functions, The Riesz-Thorin Theorem, and Strong and Weak Type

1.1 Cutoff functions

Definition 1.1. For $f : \mathbb{C} \to \mathbb{R}$, $A > 0$, the **cutoff** function $\phi_A \in C(\mathbb{C}, \mathbb{C})$ is

$$\phi_A(z) = \begin{cases} z & |z| < A \\ Az/|z| & |z| > A. \end{cases}$$

Note that $\phi_A(\mathbb{C}) = \overline{B_A(0)}$ and $\phi_A(\mathbb{R}) \subseteq \mathbb{R}$.

Theorem 1.1. *Let $f : X \to \mathbb{C}$ be measurable, and for $A > 0$, set*

$$h_A = \phi_A \circ f, \qquad g_A = f - h_A.$$

Then

$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \alpha < A \\ 0 & \alpha \geq A \end{cases}, \qquad \lambda_{g_A}(\alpha) = \lambda_f(\alpha + A).$$

Proof. Let $\alpha > 0$. Since $|h_A| \leq A$, $\{h_A > \alpha\} = \varnothing$ if $\alpha \geq A$. This shows $\lambda_{h_A}(\alpha) = 0$. If $0 < \alpha < A$, then $\{|h_A| > \alpha\} = \{|f| > \alpha\}$, so $\lambda_{h_A}(\alpha) = \lambda_f(\alpha)$.

Note that

$$g_A = f - \varphi_A \circ f \implies |g_A| = |f - \phi_A \circ f| = \begin{cases} 0 & |f| < A \\ |f - \frac{f}{|f|}A| & |f| > A. \end{cases}$$

Hence,

$$|g_A| = \begin{cases} 0 & |f| < A \\ |f| - A & |f| \geq A. \end{cases}$$

So if $\alpha > 0$, then $\{|g_A| > \alpha = \{|f| - A > \alpha\} = \{|f| < \alpha + A\}$. $\square$

1.2 The Riesz-Thorin interpolation theorem

Throughout this section (interpolation of L^p spaces), $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces.

Let $p < q < r$. If $t \geq 0$, then

$$t^q \leq \begin{cases} t^r & t \geq 1 \\ t^p & 0 \leq 0 \leq 1. \end{cases}$$

So for any $t \in \mathbb{R}$, $|t|^q \leq |t|^p + |t|^r$ for all t. Hence, if $f : X \to \mathbb{C}$ is μ-measurable, then $|f|^q \leq |f|^r + |f|^p$. We get the following.

Proposition 1.1. $L^r(\mu) \cap L^p(\mu) \subseteq L^q(\mu)$.

Recall that ν is called **semifinite** if for any $E \in \mathcal{N}$ such that $\nu(E) = \infty$, there exists $F \in \mathcal{N}$ such that $F \subseteq E$ and $0 < \nu(F) < \infty$.

Theorem 1.2 (Riesz-Thorin interpolation theorem)**.** *Let $1 \leq p_0, q_0, p_1, q_1 < \infty$, and further assume that ν is semifinite if $q_0 = q_1 = \infty$. For $t \in (0,1)$, define p_t and q_t as*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Assume $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to L^{q_0}(\nu) + L^{q_1}(\nu)$ is a linear operator such that there are M_0,$M_1 \geq 0$ such that

$$\|Tf\|_{L^{q_0}(\nu)} \leq M_0 \|f\|_{L^{p_0}(\mu)}, \qquad \|Tg\|_{L^{q_1}(\nu)} \leq M_1 \|g\|_{L^{p_1}(\nu)}$$

for every $f \in L^{p_0}(\mu)$ and $g \in L^{p_1}(\mu)$. Then

$$\|Th\|_{L^{q_t}(\nu)} \leq M_0^{1-t} M_1^t \|h\|_{L^{p_t}(\mu)}$$

for all $h \in L^{p_t}(\mu)$.

Remark 1.1. It it not surprising that this is bounded. The particular bound is the important part.

$$|Th|^{q_t} \leq |Th|^{q_0} + |Th|^{q_1},$$

so

$$\|Th\|_{L^{q_t}}^{q_t} \leq \|Th\|_{L^{q_0}}^{q_0} + \|Th\|_{L^{q_1}}^{q_1} \leq M_0^{q_0} \|h\|_{L^{p_0}}^{q_0} + M_1^{q_1} \|h\|_{L^{p_1}}^{q_1}.$$

We will not prove this theorem, as it involves a lemma that is technical and not very instructive.

1.3 Strong type and weak type

Let $\mathcal{D}$ be a vector subspace of the set of $(X, \mathcal{M}, \mu)$ measurable functions, and let $\mathcal{F}$ be the set of $(Y, \mathcal{N}, \nu)$ measurable functions.

Definition 1.2. We say that $T : \mathcal{D} \to F$ os sublinear if

1. $|T(f+g)| \leq |Tf| + |Tg|$
2. $|T(cf)| = c|Tf|$

for all $f, g \in \mathcal{D}$ and $c \geq 0$.

Definition 1.3. Let $T : \mathcal{D} \to \mathcal{F}$ be a sublinear map, and let $1 \leq p, q \leq \infty$. We say that T is (p,q)**-strong type** if there exists $c > 0$ such that

$$\|Tf\|_{L^q} \leq c\|f\|_{L^p}$$

for all $f \in \mathcal{D}$. We say that T is (p,q)**-weak type** if there exists $c > 0$ such that

$$[Tf]_q \leq c\|f\|_{L^p}$$

for all $f \in \mathcal{D}$, provided that $q < \infty$. We say that T is (p, ∞)**-weak type** if T is (p, ∞)-strong type.

Remark 1.2. If $f \in \mathcal{D}$ but $f \notin L^p(\mu)$, then the right hand side is ∞, satisfying the inequality. So we could replace the condition with $f \in L^p(\mu)$.

We can rewrite the strong type condition as

$$q \int_0^\infty \alpha^{q-1} \lambda_{T(f)}(\alpha)\, d\alpha \leq c^q \left(p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, d\alpha \right)^{q/p}.$$

Proposition 1.2. *Assume $f : X \to \mathbb{C}$ is measurable and $1 \leq p < \infty$. Then*

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, \alpha.$$

Proof. If $f \in L^p(\mu)$, we have already proved this. Otherwise, suppose $f = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$, where the A_i are measurable and pairwise disjoint. Since

$$\infty = \|f\|_{L^p}^p = \sum_{i=1}^n |a_i|^p \mu(A_i),$$

there is i such that $\mu(A_i) = \infty$. We have $\lambda_f \geq \lambda_{|a_i|\mathbb{1}_{A_i}}$. But $\lambda_{|a_i|\mathbb{1}_{A_i}}(\alpha) = \infty$ if $\alpha \in (0, |a_i|)$. Now for general f, approximated it form below by step functions and apply the dominated convergence theorem. □